Probability

PSY 5101: Advanced Statistics for Psychological and Behavioral Research 1

Probability

- The classic theory of probability underlies much of probability in statistics
- It states that the chance of a particular outcome occurring is determined by the ratio of the number of favorable outcomes (or "successes") to the total number of outcomes
- Expressed as a formula:

p(A) = $\frac{Number\ of\ favorable\ outcomes}{Total\ number\ of\ possible\ outcomes}$

Probability

- What is the probability of drawing the ace of spades from a deck of cards?
  - How many ace of spades in a deck?
    1
  - How many cards in a deck?
    52
  - Probability of drawing an ace on a single draw is:
    $\frac{1}{52}$ = .0192

Probability

- What is the probability of drawing an ace (of any kind) from a deck of cards?
  - How many aces in a deck?
    4
  - How many cards in a deck?
    52
  - Probability of drawing an ace on a single draw is:
    $\frac{4}{52} = .077$
- In statistical analysis, probability is often expressed as a decimal and ranges from 0 (no chance) to a high of 1.0 (certainty)
  - Always between 0 and 1
  - Never negative

Example: The Great Shazam

- The Great Shazam claims that he can telekinetically influence coin flips (but he admits that his power is not perfect)
- You test him by asking him to predict the results of a coin that you flip
- He correctly calls heads…are you convinced of his ability?
  - Probably not…50% chance of being correct on a single coin toss
- Let's imagine that he agrees to predict 10 coin tosses (for the sake of simplicity, he always predicts heads)
- Would we be impressed if he was correct for 5 of the 10 trials?
  - No, because that is the expected frequency (.5 probability x 10 trials)
- How many would he need to get correct in order to be impressed?
  - There are 1,024 possible patterns of 10 coin toss results (2^{10})
  - There is 1 pattern with 10 heads ($\frac{1}{1,024} = .001$)
  - There are 11 patterns with 9 or more heads ($\frac{11}{1,024} = .011$)
- An extreme claim such as this would likely require considerable evidence (probably more than 10 coin tosses)
  - We would likely expect some sort of trickery on his part

Probability and Inferential Statistics

- The logic of the Great Shazam example is similar to what is used for almost all inferential statistics
- First, a researcher makes a set of observations
- Second, these observations are compared with what we would expect to observe if nothing unusual was happening in the experiment (under conditions when the null hypothesis is true)
  - This comparison is converted to a probability (i.e., the probability that the observed results would have emerged if the null hypothesis was true)
- Third, if this probability is sufficiently low, then we conclude that the alternative hypothesis is probably correct

Probability

- Probability is defined as relative frequency of occurrence
- Basic definitions:
  - Sample space: all possible outcomes of an experiment
  - Elementary event: a single member of the sample space
  - Event: any collection of elementary events
  - Probability:
    - p(elementary event) = $\frac{1}{Total\ number}$
    - p(event) = $\frac{Number\ in\ the\ event}{Total\ number}$
  - Conditional probability:
    - p(A|B) = $\frac{Number\ in\ (A\ and\ B)}{Number\ in\ B}$
    - The probability of A in the redefined (reduced) sample space of B

Probability: The Juror Example

	Authoritarian	Egalitarian	Totals
Award	18	13	31
No Award	2	15	17
Totals	20	28	48

- The sample space is all 48 jurors
- An elementary event is any one of the 48 jurors
- An event is any subgroup of the 48 (e.g., the 31 who gave an award)
- Probabilities:
  - p(elementary event) = $\frac{1}{48}$
  - p(award) = $\frac{31}{48}$ = .65
- Conditional probability: p(Award|Authoritarian) = $\frac{18}{20}$
  - The 20 Authoritarians are the reduced sample space
  - ***Always*** do the denominator first
  - Out of the 20 Authoritarians, the 18 who gave an award go in the numerator

Probability: The Big Three (+1)

1. Independence: events A and B are independent if
- p(A)=p(A|B)
- The A probability is not changed by reducing the sample space to B OR the occurrence of one event does not change the probability of occurrence of another event

2. Multiplication (And) Rule: the probability of a joint occurrence
- p(A and B)=p(A)p(B|A)=p(A|B)p(B)

- Mutually exclusive:
  - Events A and B do not have any elementary events in common
  - Events A and B cannot occur simultaneously
  - p(A and B)=0

3. Addition (Or) Rule: include all cases in one event or the other
- p(A or B)=p(A)+p(B)-p(A and B)

Independence

	Authoritarian	Egalitarian	Totals
Award	18	13	31
No Award	2	15	17
Totals	20	28	48

- The first of the Big 3: Independence is p(A)=p(A|B)
  - Is Award independent of Authoritarian? It is if p(Award) = p(Award|Authoritarian)
  - p(Award) = $\frac{31}{48}$ =.65
  - p(Award|Authoritarian) = $\frac{18}{20}$ = .90
  - No, Award is not independent of Authoritarian because .65≠.90
- Another example
  - Is No Award independent of Egalitarian?
  - It is if p(No Award) = p(No Award|Egalitarian)
  - p(No Award) = $\frac{17}{48}$ = .35
  - p(No Award|Egalitarian) = $\frac{15}{28}$ = .54
  - No, No Award is not independent of Egalitarian because .35≠.54

More About Independence

- Independence is p(A)=p(A|B) or p(B)=p(B|A):
  - A is independent of B if p(A)=p(A|B) or if p(B)=p(B|A)
- Examples:
  - If p(A)=.3 and p(A|B)=.4, A and B are not independent because .3≠.4
  - Given: p(A)=.1, p(B)=.2, p(A|B)=.3, and p(B|A)=.4. Is A independent of B? Explain

    No, A and B are not independent because .1≠.3 (or .2≠.4)
  - Given: p(A)=.1, p(B)=.2, p(A|B)=.1, and p(B|A)=.2. Is A independent of B? Explain

    Yes, A and B are independent because .1=.1 (or .2=.2)
  - Given: p(A)=.47, p(B)=.34, and p(B|A)=.49. Which of these is the reason that A is not independent of B?

    .47≠.34, .47≠.49, .34≠.49

Answer: .34≠.49

More About Independence

	Before Dinner	After Dinner	Totals
Dessert	34	21	55
No Dessert	18	31	49
Totals	52	52	104

- People in a restaurant were asked before and after their meal if they thought they would like a dessert. Is Dessert independent of Before?
  - p(Dessert) = $\frac{55}{104}$ =.53 and p(Dessert|Before) = $\frac{34}{52}$ =.65
  - No, Dessert is not independent of Before because .53≠.65,
  - OR p(Before) = $\frac{52}{104}$ =.50 and p(Before|Dessert) = $\frac{34}{55}$ = .62
  - No, Dessert is not independent of Before because .50≠.62
- Which two probabilities would you have to examine to determine if No Dessert is independent of After?
  - One way: see if p(No Dessert) equals p(No Dessert|After)
  - Or, see if p(After) equals p(After|No Dessert)

Multiplication Rule

	Authoritarian	Egalitarian	Totals
Award	18	13	31
No Award	2	15	17
Totals	20	28	48

- The second of the Big 3: Multiplication (And) Rule: p(A and B)=p(A)p(B|A)= p(B)p(A|B)
  - This is the product of two probabilities: one about A, one about B, one a marginal probability, one a conditional probability.
  - Compute p(Award and Authoritarian). We know the answer to this is $\frac{18}{48}$ = .375
  - P(Award and Authoritarian)=p(Award)p(Authoritarian|Award) = $\frac{31}{48}*\frac{18}{31}=\frac{18}{48}$
    - p(Award and Authoritarian)=p(Auth)p(Award|Authoritarian) = $\frac{20}{48}*\frac{18}{20}=\frac{18}{48}$
  - Compute p(Award and Egalitarian). We know the answer to this is $\frac{13}{48}$ = .27
    - p(Award and Egalitarian) = p(Award)p(Egalitarian|Award)= $\frac{31}{48}*\frac{13}{31}=\frac{13}{48}$
    - p(Award and Egalitarian)=p(Egalitarian)p(Award|Egalitarian)= $\frac{28}{48}*\frac{13}{28}=\frac{13}{48}$

More About Multiplication Rule

- Multiplication (And) Rule: p(A and B)=p(A)p(B|A)= p(B)p(A|B)
  - This is the product of two probabilities: one about A, one about B, one a marginal probability, one a conditional probability
  - Other examples:
    - p(A)=.2, p(B)=.6, p(A|B)=.3, p(A and B)= p(B)p(A|B)=(.6)(.3)=.18
    - p(A)=.5, p(B)=.1, p(B|A)=.2, p(A and B) =p(A)p(B|A)=(.5)(.2)=.10
  - If all four probabilities are given, you can do the problem two ways and you should get the same answer: p(A)=.2, p(B)=.4, p(A|B)=.3, p(B|A)=.6,
    - p(A and B)=p(A)p(B|A)=(.2)(.6)=.12
    - p(A and B)=p(B)p(A|B)=(.4)(.3)=.12

Mutually Exclusive

	Authoritarian	Egalitarian	Totals
Award	18	13	31
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- Not one of the Big 3 and not the same as independence
- Are No Award and Authoritarian mutually exclusive?
  - No, because p(No Award and Authoritarian) = $\frac{2}{48} \neq 0$
  - If none of the 48 jurors had responded in this cell, then No Award and Authoritarian would have been mutually exclusive
- Are Award and No Award mutually exclusive? Yes, because p(Award and No Award)=0

Addition Rule

	Authoritarian	Egalitarian	Totals
Award	18	13	31
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- The third of the Big 3: Addition (Or) Rule: p(A or B)= p(A) + p(B) - p(A and B)
  - This is the most complicated of the Big 3 because it uses the Multiplication Rule to get its answer
  - Compute p(Award or Authoritarian)
    - p(Award and Authoritarian) = p(Award)*p(Authoritarian|Award) = $\frac{31}{48}*\frac{18}{31}=\frac{18}{48}$
    - p(Award or Authoritarian)= p(Award)+p(Authoritarian)-p(Award and Authoritarian) = $\frac{31}{48}+\frac{20}{48}-\frac{18}{48}=\frac{33}{48}$ = .69
  - Compute p(Award or Egalitarian)
    - p(Award and Egalitarian) = p(Award)*p(Egalitarian|Award) = $\frac{31}{48}*\frac{13}{31}=\frac{13}{48}$
    - p(Award or Egalitarian)= p(Award)+p(Egalitarian)-p(Award and Egalitarian) = $\frac{31}{48}+\frac{28}{48}-\frac{13}{48}=\frac{46}{48}$ = .96

More About the Addition Rule

- Addition (Or) Rule: p(A or B)=p(A)+p(B)-p(A and B)
  - Note that you add together two marginal probabilities and subtract off a joint (And) probability
- Other examples: find p(A or B)
  - p(A)=.2, p(B)=.6, p(A|B)=.3
    - First, compute p(A and B): p(A and B) = p(B)p(A|B)=(.6)(.3)=.18
    - Now, compute p(A or B): p(A or B) = p(A)+p(B)-p(A and B) = .2+.6-.18=.62
  - p(A)=.5, p(B)=.1, p(B|A)=.2
    - First, compute p(A and B): p(A and B) = p(A)p(B|A)=(.5)(.2)=.10
    - Now, compute p(A or B): p(A or B) = p(A)+p(B)-p(A and B) = .5+.1-.10=.50

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