Two-Way ANOVA

PSY 5101: Advanced Statistics for Psychological and Behavioral Research 1

Goals

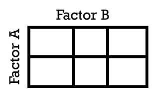
- Rationale for Factorial ANOVA
- Partitioning Variance
- Interaction Effects
  - Interaction Graphs
  - Interpretation

What is Two-Way ANOVA?

- Two Factors (i.e., variables that classify participants into groups)
  - Two-way = 2 factors
  - Three-way = 3 factors
- For now, we are going to focus on situations in which there are different participants in *all* conditions
  - This should be used with between-subjects designs
- More than one factor is known as a "factorial design"
- Later, we will talk about repeated-measures designs (same participants in all conditions) and mixed designs (blend of between-subjects and within-subjects conditions)

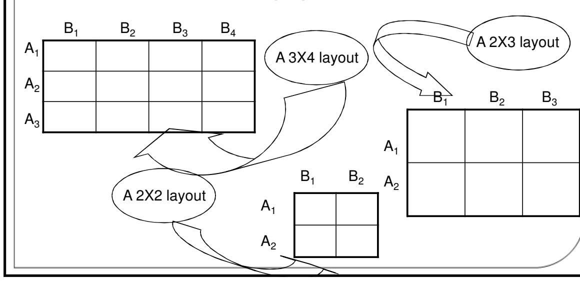
Two-Way ANOVA: Introduction

- The two-way ANOVA uses two factors, variables that combine to form the groups
  - The factors may or may not be independent variables
- The groups formed by combining levels/values of the factors are called cells, and the means of the observations in these cells are called "cell means"
- We have three F-tests in a two-way ANOVA, one for each of the two factors by themselves, and one for the interaction of the two factors



Two-Way ANOVA: Factors *vs.* Levels

- Each two-way ANOVA always has two factors, but each of these factors can have different numbers of levels (levels are the values of the factors)
- Here are some different two-way layouts:



Two-Way ANOVA: Logic

- The logic of the two-way ANOVA is the same as that for the one-way:
  - For each of the three F-tests, you will form an F-ratio based on two variances
  - For each F, if H_o is true, both variances should be equal and the average F will be about 1
- For each F, if H_0 is false:
  - We expect numerator > denominator
  - We expect average F > 1
  - And we reject H_o if $F \geq F_{crit}$
- The difference is that the two-way ANOVA is more complex because there are three F-ratios
  - The effects of the factors are called main effects (and the F for the interaction)

Two-Way ANOVA: F-tests

Two-way ANOVA F-tests

1. Situation/hypotheses	Two factors, J levels of A, K levels of B, n observations per cell For A, $H_o: \mu_1 = \mu_2 = \ldots = \mu_J$ For B, $H_o: \mu_1 = \mu_2 = \ldots = \mu_K$ For AxB, H_o: no interaction effect
2. Test statistic	$F_A = \frac{MS_A}{MS_{Within}}$ $F_B = \frac{MS_B}{MS_{Within}}$ $F_{AxB} = \frac{MS_{AxB}}{MS_{Within}}$
3 .Distribution	$F_{J-1,\ JK(n-1)}$ $F_{K-1,\ JK(n-1)}$ $F_{(J-1)(K-1),\ JK(n-1)}$
4. Assumptions	1. Populations are normal 2. Equal population variances for each cell 3. Observations are independent

Similar to One-Way ANOVA

Benefit of Factorial Designs

- We can look at how variables **interact**
- Interactions
  - Show how the effects of one factor might depend on the effects of another factor
  - Interactions indicate "moderation" effects
  - Interactions are often more interesting than main effects
- Example
  - There may be an interaction between hangover and lecture topic on sleeping during class
    - A hangover might have more effect on sleepiness during a stats lecture than during a lecture about sexual behavior

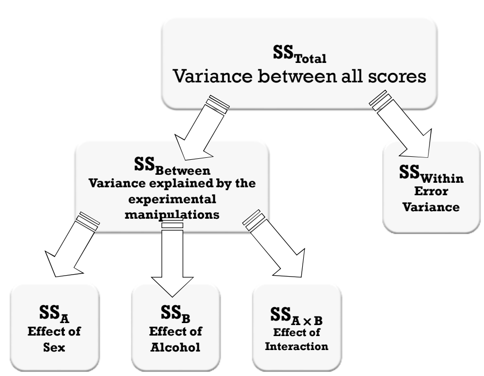
An Example

- The effects of Alcohol and Sex on "the beer-goggles effect" conducted by an anthropologist
  - Phenomenon in which drinking alcohol increases the perceived attractiveness of others in the social environment (summed up by the phrase "There are no ugly women at closing time")
- Factors
  - Factor A (Sex): Male, Female
  - Factor B (Alcohol): None, 2 pints of lager, 4 pints of lager
- Outcome variable was an objective measure of the attractiveness of the partner selected at the end of the evening
  - Took a photo of the person the participant was speaking to at a designated time and had independent judges rate the attractiveness of the person

Two-Way ANOVA: Logic

- Notation:
  - n = number of observations per cell
  - J = number of levels of Factor A
  - K = number of levels of Factor B
  - N = total number of participants (nJK)
- Each of the three F's is formed as a ratio of two sample variances: the numerator will be the MS for the effect tested (MS_A, MS_B, or MS_{AxB}), the denominator will be MS_W
- Hypotheses:
  - For A (e.g., Sex)
    - H_o: $\mu_1=\mu_2=\ldots=\mu_J$
    - H_1: any differences in μ_js
  - For B (e.g., Alcohol)
    - H_o: $\mu_1=\mu_2=\ldots=\mu_K$
    - H_1: any differences in μ_ks
  - For interaction (not easily expressed in terms of μs)
    - H_o: no interaction effect
    - H_1: some interaction effect

Alcohol	**None**		**2 Pints**		**4 Pints**	
Sex	Female	Male	Female	Male	Female	Male
	65	50	70	45	55	30
	50	55	65	60	65	30
	70	80	60	85	70	30
	45	65	70	65	55	55
	55	70	65	70	55	35
	30	75	60	70	60	20
	70	75	60	80	50	45
	55	65	50	60	50	40
Total	485	535	500	535	460	285
Mean	60.625	66.875	62.50	66.875	57.50	35.625
Variance	**24.55**	**106.70**	**42.86**	**156.70**	**50.00**	**117.41**



Step 1: Calculate SS_{Total}

65	50	70	45	55	30
50	55	65	60	65	30
70	80	60	85	70	30
45	65	70	65	55	55
55	70	65	70	55	35
30	75	60	70	60	20
70	75	60	80	50	45
55	65	50	60	50	40

Grand Mean = 58.33

This is the total amount of variability that can be explained

$$\begin{aligned} SS_{Total} &= s^2_{grand}(N-1) \\ &= 190.78(48-1) \\ &= 8966.66 \end{aligned}$$

Step 2: Calculate $SS_{Between}$

$$SS_{Between} = \sum n_i\left(\bar{x}_i - \bar{x}_{grand}\right)^2$$

$$\begin{aligned} SS_{Between} &= 8(60.625-58.33)^2 + 8(66.875-58.33)^2 + 8(62.5-58.33)^2 \\ &\quad + 8(66.875-58.33)^2 + 8(57.5-58.33)^2 + 8(35.625-58.33)^2 \\ &= 8(2.295)^2 + 8(8.545)^2 + 8(4.17)^2 + 8(8.545)^2 + 8(-0.83)^2 + 8(-22.705)^2 \\ &= 42.1362 + 584.1362 + 139.1112 + 584.1362 + 5.5112 + 4124.1362 \\ &= 5479.167 \end{aligned}$$

This is the amount of variability that is explained by the factors

Step 2a: Calculate SS_A

A_1: Female		
65	70	55
70	65	65
60	60	70
60	70	55
60	65	55
55	60	60
60	60	50
55	50	50

Mean Female = 60.21

A_2: Male		
50	45	30
55	60	30
80	85	30
65	65	55
70	70	35
75	70	20
75	80	45
65	60	40

Mean Male = 56.46

$$SS_A = \sum n_j\left(\bar{x}_j - \bar{x}_{grand}\right)^2$$

$$\begin{aligned} SS_{Sex} &= 24(60.21-58.33)^2 + 24(56.46-58.33)^2 \\ &= 24(1.88)^2 + 24(-1.87)^2 \\ &= 84.8256 + 83.9256 \\ &= 168.75 \end{aligned}$$

This is the amount of variability that is explained by Factor A (Sex)

Step 2b: Calculate SS_B

B_1: None		B_2: 2 Pints		B_3: 4 Pints	
65	50	70	45	55	30
70	55	65	60	65	30
60	80	60	85	70	30
60	65	70	65	55	55
60	70	65	70	55	35
55	75	60	70	60	20
60	75	60	80	50	45
55	65	50	60	50	40

Mean None = 63.75 **Mean 2 Pints = 64.6875** **Mean 4 Pints = 46.5625**

$$SS_B = \sum n_k\left(\bar{x}_k - \bar{x}_{grand}\right)^2$$

$$\begin{aligned} SS_{alcohol} &= 16(63.75-58.33)^2 + 16(64.6875-58.33)^2 + 16(46.5625-58.33)^2 \\ &= 16(5.42)^2 + 16(6.3575)^2 + 16(-11.7675)^2 \\ &= 470.0224 + 646.6849 + 2215.5849 \\ &= 3332.292 \end{aligned}$$

This is the amount of variability that is explained by Factor B (Alcohol)

Step 2c: Calculate $SS_{A \times B}$

$$SS_{A\times B} = SS_{Between} - SS_A - SS_B$$

$$\begin{aligned} SS_{A\times B} &= SS_{Between} - SS_A - SS_B \\ &= 5479.167 - 168.75 - 3332.292 \\ &= 1978.125 \end{aligned}$$

This is the amount of variability that is explained by the interaction of Factor A (Sex) and Factor B (Alcohol)

Step 3: Calculate SS_{Within}

$$SS_{Within} = s^2_{group1}(n_1-1) + s^2_{group2}(n_2-1) + s^2_{group3}(n_3-1) \ldots s^2_{group\,n}(n_n-1)$$

$$\begin{aligned} SS_{Within} &= s^2_{group1}(n_1-1) + s^2_{group2}(n_2-1) + s^2_{group3}(n_3-1) \\ &\quad + s^2_{group4}(n_4-1) + s^2_{group5}(n_5-1) + s^2_{group6}(n_6-1) \\ &= (24.55\times 7) + (106.7\times 7) + (42.86\times 7) \\ &\quad + (156.7\times 7) + (50\times 7) + (117.41\times 7) \\ &= 171.85 + 746.9 + 300 + 1096.9 + 350 + 821.87 \\ &= 3487.52 \end{aligned}$$

This is the amount of variability that is not explained by Factor A (Sex), Factor B (Alcohol), or their interaction

Degrees of Freedom for Two-Way ANOVA

- J = number of levels of Factor A
- K = number of levels of Factor B
- $df_A = J - 1 = 2 - 1 = 1$
- $df_B = K - 1 = 3 - 1 = 2$
- $df_{A \times B} = (J - 1)*(K - 1) = (2 - 1)*(3 - 1) = 1 * 2 = 2$
- $df_{Within} = JK(n-1) = 6*7 = 42$

Mean Squares for Two-Way ANOVA

- $MS_A = \frac{SS_A}{df_A} = \frac{168.75}{1} = 168.75$
- $MS_B = \frac{SS_B}{df_B} = \frac{3332.292}{2} = 1666.146$
- $MS_{A \times B} = \frac{SS_{A \times B}}{df_{A \times B}} = \frac{1978.125}{2} = 989.062$
- $MS_{Within} = \frac{SS_{Within}}{df_{Within}} = \frac{3487.52}{42} = 83.036$

F-Ratios for Two-Way ANOVA

- $F_A = \frac{MS_A}{MS_{Within}} = \frac{168.75}{83.036} = 2.032$
- $F_B = \frac{MS_B}{MS_{Within}} = \frac{1666.146}{83.036} = 20.065$
- $F_{A \times B} = \frac{MS_{A \times B}}{MS_{Within}} = \frac{989.062}{83.036} = 11.911$

Summary Table

Tests of Between-Subjects Effects

Dependent Variable: Attractiveness of Date

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	5479.167^a	5	1095.833	13.197	.000
Intercept	163333.333	1	163333.333	1967.025	.000
Gender	168.750	1	168.750	2.032	.161
Alcohol	3332.292	2	1666.146	20.065	.000
Gender * Alcohol	1978.125	2	989.062	11.911	.000
Error	3487.500	42	83.036		
Total	172300.000	48			
Corrected Total	8966.667	47			

a. R Squared = .611 (Adjusted R Squared = .565)

This is the main effect of Gender

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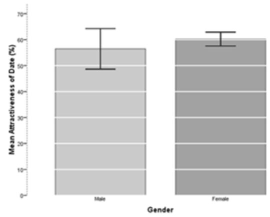
Error	3487.500	42	83.036		
Total	172300.000	48			
Corrected Total	8966.667	47			

a. R Squared = .611 (Adjusted R Squared = .565)

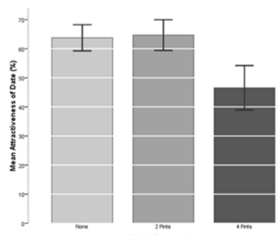
This is the Interaction

Interpretation: Main Effect of Sex



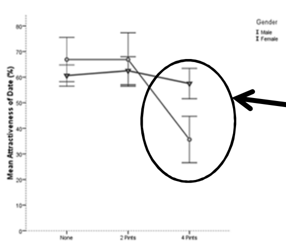
There was not a significant main effect of sex for the attractiveness of selected mates, $F(1, 42) = 2.03$, $p = .161$.

Interpretation: Main Effect of Alcohol



There was a significant main effect of the amount of alcohol consumed at the night-club on the attractiveness of the mate that was selected, $F(2, 42) = 20.07$, $p < .001$.

Interpretation: Interaction



It appears that the interaction between sex and alcohol is due to men selecting less attractive mates after consuming larger amounts of alcohol…but this would need to be confirmed using post-hoc tests

There was a significant interaction between the amount of alcohol consumed and the sex of the person selecting a mate on the attractiveness of the partner selected, $F(2, 42) = 11.91$, $p < .001$.

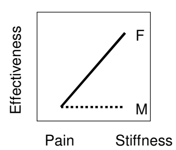
Interpreting the Two-Way ANOVA Plot

1. If the lines are not parallel, then an interaction is indicated (may or may not be significant depending on chance variability)
2. If the midpoints of the lines are not equal, then a main effect of Factor A is indicated
3. If the visual average (middle) of the points (cell means) above each level of Factor B are not equal, then a main effect of Factor B is indicated

Two-Way ANOVA: Interaction

- Symptoms of arthritis frequently include stiffness and joint pain
- A new drug helps only women who experience stiffness, not women with joint pain nor men with either symptom
- Using a rating of the drug that increases as effectiveness of the drug increases, these results would look like this:

- Gender (M or F) is interacting with symptom (Stiffness or Joint Pain)



Two-Way ANOVA: Interaction

- Plots of cell means showing the three F-tests (assumes that MS_W is small so any observed difference is significant).

	B_1	B_2	
A_1	50	50	50
A_2	50	50	50
	50	50	

F_A: NS
F_B: NS
F_{AB}: NS

	B_1	B_2	
A_1	60	40	50
A_2	60	40	50
	60	40	

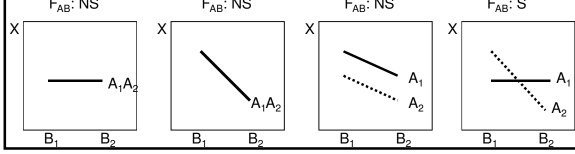
F_A: NS
F_B: S
F_{AB}: NS

	B_1	B_2	
A_1	60	50	55
A_2	50	40	45
	55	45	

F_A: S
F_B: S
F_{AB}: NS

	B_1	B_2	
A_1	50	50	50
A_2	60	40	50
	55	45	

F_A: NS
F_B: S
F_{AB}: S



Two-Way ANOVA: Interaction

- Plots of cell means showing the three F-tests (assumes that MS_W is small so any observed difference is significant).

	B_1	B_2	
A_1	55	55	55
A_2	45	45	45
	50	50	

F_A: S
F_B: NS
F_{AB}: NS

	B_1	B_2	
A_1	60	40	50
A_2	40	60	50
	50	50	

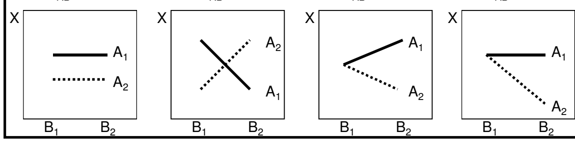
F_A: NS
F_B: NS
F_{AB}: S

	B_1	B_2	
A_1	50	60	55
A_2	50	40	45
	50	50	

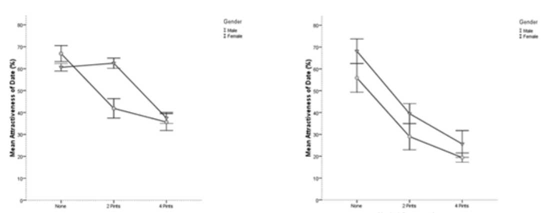
F_A: S
F_B: NS
F_{AB}: S

	B_1	B_2	
A_1	55	55	55
A_2	55	35	45
	55	45	

F_A: S
F_B: S
F_{AB}: S

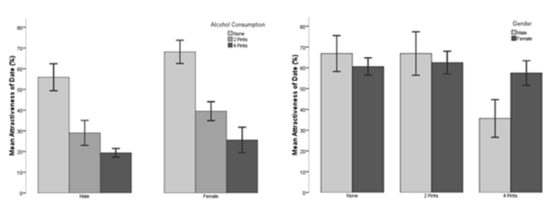


Is there likely to be a significant interaction effect?



Yes **No**

Is there likely to be a significant interaction effect?



No **Yes**